



The near wall mixing length formulation revisited

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1. Introduction

The calculation of transport phenomena in boundary layers, pipe or duct flows can be easily accomplished by a simple, zero order, mixing length formulation. The application of this model in simple engineering flows, i.e., flows without recirculation or asymmetry, is still widely extended because of its simplicity and its ability to match experimental results at the same degree than more elaborated models. Most of the mixing-length formulations for near-wall flows use a damping factor to decrease rapidly the characteristic length as the wall is approached. The most popular damping factor is the one suggested by van Driest [1], which has been modified extensively to accommodate different wall conditions [2–6].

A yet unresolved and apparently intrinsic shortcoming of the mixing length model is that it cannot predict properly the turbulent viscosity profile ($\nu_t \propto y^3$) in the near-wall region. The purpose of the current study is to demonstrate that the mixing length theory can be formulated consistently with available information of the turbulent viscosity profiles near the wall and to propose a mixing length equation valid for momentum and heat transfer across turbulent wall-bounded flows. This formulation could also improve large-eddy simulations (LES) carried out with subgrid models, where the length scale in the direction normal to the wall is

modified with a damping factor to improve numerical predictions (see for example [7]).

2. Mixing length equation

The turbulent or eddy viscosity, ν_t , very near the wall can be calculated according to

$$\frac{\nu_t}{\nu} = by^{+n} \quad (1)$$

where ν is the molecular viscosity, b and n are constants, y is the distance from the wall and the superscript $+$ indicates normalization with respect to friction velocity and ν . It has been well established that $n = 3$ [8–13] and $0.0009 \leq b \leq 0.001$ [8,10,12]. Eq. (1) with $n = 3$ was first derived by Murphree [14] applying asymptote expansions.

The turbulent viscosity can be expressed in terms of the mixing length l and expanded around $y^+ = 0$ using a MacLaurin series,

$$\begin{aligned} \frac{\nu_t}{\nu} = l^{+2} \left| \frac{du^+}{dy^+} \right| &= \left(l_w^{+'} \right)^2 y^{+2} + l_w^{+'} l_w^{+''} y^{+3} \\ &+ \left(\frac{l_w^{+'} l_w^{+'''} }{3} + \frac{(l_w^{+''})^2}{4} \right) y^{+4} + O(y^{+5}) \end{aligned} \quad (2)$$

where the subscript 'w' identifies wall values, and the prime differentiation with respect to y^+ . It has been assumed that $du^+/dy^+ = 1$ near the wall ($y^+ < 5$) and that $l_w = 0$ to cancel the turbulent contribution to the total viscosity at the wall. The series approximation (2) requires all derivatives to be finite at $y = 0$.

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When l^+ is formulated with the van Driest viscous damping [1] of the linear mixing length variation with wall distance,

$$l^+ = \kappa y^+ [1 - \exp(-y^+/A)] \tag{3}$$

the first non-zero coefficient in Eq. (2) corresponds to the term y^{+4} because $l'_w = 0$. In Eq. (3), $A = 26$ and κ is the von Karman constant. Any formulation with $l'_w \neq 0$, such as the modification of the van Driest exponential damping function proposed by Grifoll and Giralt [15], to predict high Schmidt number mass transfer coefficients, results in $n = 2$.

The mixing length model as formulated in Eq. (2) seems incapable of reproducing the y^{+3} variation near the wall. When the original van Driest's constant A in Eq. (3) is modified according to

$$A = A_0 [1 - \exp(-y^+/C)]^{1/2} \tag{4}$$

the first derivative of the mixing length in Eq. (2) remains zero at the wall, $l'_w = 0$, while $l''_w \rightarrow \infty$ as $y^+ \rightarrow 0$, making Eq. (2) not applicable. Nevertheless, a series expansion can still be derived if the square root of y^+ is used as variable to force all derivatives to be finite at $y = 0$. In this case, the series expansion becomes,

$$\frac{v_t}{\nu} = \frac{C\kappa^2}{A_0^2} y^{+3} - \frac{C^{3/2}\kappa^2}{A_0^3} y^{+7/2} + O(y^{+4}) \tag{5}$$

The comparison of Eq. (5) for $y^+ \rightarrow 0$ and Eq. (1) with $n = 3$ yields

$$C = \frac{bA_0^2}{\kappa^2} \tag{6}$$

For the value $b = 0.001$ suggested by Kays [10] and

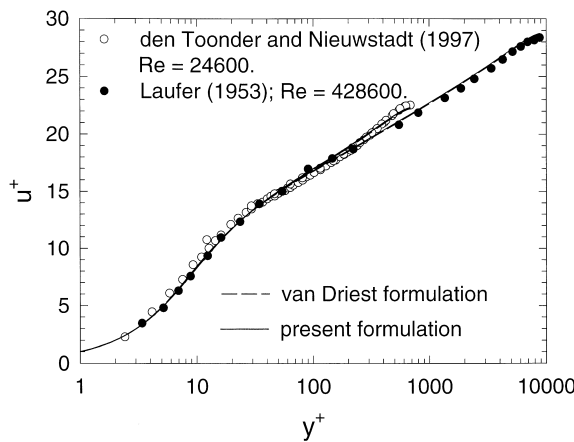


Fig. 1. Experimental and calculated dimensionless velocity profiles.

$\kappa = 0.4$, the best match between the velocity distribution for pipe flow predicted by the present mixing length equations (3) and (4), subject to Eq. (6), and by the original van Driest equation ($A = 26$) for $y^+ > 10$, is obtained with the pair of constants $C = 4.8$ and $A_0 = 27.8$. This agreement is illustrated in Fig. 1, where both predicted velocity profiles are compared with data measured by den Toonder et al. [16] at $Re = 24,600$ and by Laufer [17] at $Re = 428,600$. The maximum difference between the two calculated velocity distributions is less than 1.5% at $y^+ \sim 9$, with an average difference less than 0.5%. This low discrepancy is maintained along the range where the mixing length equation is applicable, i.e. $Re \geq 10,000$ [2]. The mixing length calculations of the velocity profiles in Fig. 1 were carried out with the complete Nikuradse [18] polynomial extension of Eq. (3).

The near-wall turbulent shear stress measured by den Toonder et al. [16], predicted by DNS [11,13], and calculated from the original van Driest and the present formulations of mixing length model are shown in Fig. 2. As mentioned before, the van Driest formulation predicts a y^{+4} dependency near the wall and deviates progressively from the data as the wall is approached. The present formulation agrees with DNS results and with the limited experimental data available in the near-wall region.

3. Heat transfer

Heat and/or mass transfer in simple flow geometries, where the adoption of the mixing length model for momentum transport is reasonable, can also be adequately predicted by solving the energy equation. This requires the use of a turbulent thermal diffusivity (α_t) analogous to the turbulent viscosity, corrected by a

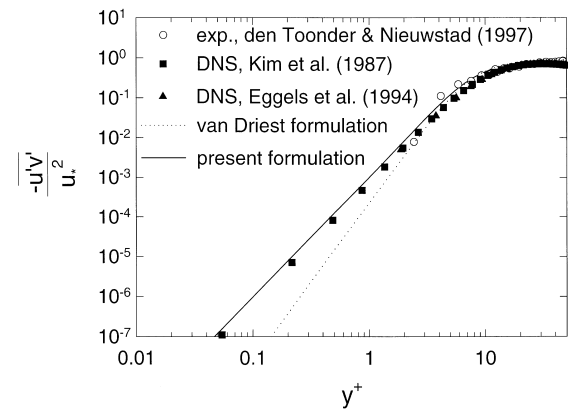


Fig. 2. Distribution of the near-wall turbulent shear stress in pipe flow.

turbulent Prandtl number, $Pr_t = \nu_t/\alpha_t$, which has to be estimated.

For pipe flow the energy equation is

$$\rho C_p u \frac{\partial T}{\partial z} = -\frac{l}{r} \frac{\partial(rq)}{\partial r} \tag{7}$$

where ρ is the density, C_p the heat capacity, u the velocity, T the temperature, z and r are axial and radial coordinates, respectively, and q is the radial heat flux given by

$$q = -\rho C_p \left(\frac{\nu_t}{Pr_t} + \frac{\nu}{Pr} \right) \frac{\partial T}{\partial r} \tag{8}$$

Direct measurements of the Pr_t are relatively scarce and exhibit a large scatter. Experimental data and DNS calculations suggest that for fully developed pipe flow and moderate to high Prandtl number conditions the Pr_t is almost independent of wall distance. Moreover, from analysis of experimental velocity and temperature profiles, Kays [10] concluded that $Pr_t = 0.85$ in the region where both profiles are logarithmic.

The energy conservation equation (7) with heat fluxes given by Eq. (9) and boundary conditions

$$T = T_w; \quad @ r = R \tag{9}$$

and

$$\frac{\partial T}{\partial r} = 0; \quad @ r = 0 \tag{10}$$

has been solved for different Reynolds and Prandtl numbers, subjected to $\partial T/\partial z = \text{constant}$, as the fully developed temperature condition requires [5].

Fig. 3 shows the variation of the Nusselt number with Prandtl number predicted with the van Driest and present mixing length formulations with a constant

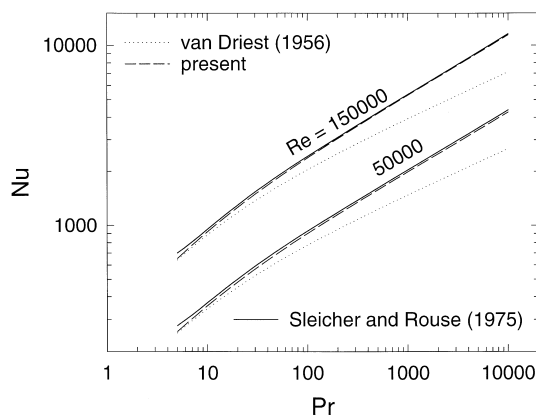


Fig. 3. Variation of the dimensionless heat transfer rates coefficients with Pr in pipe flow.

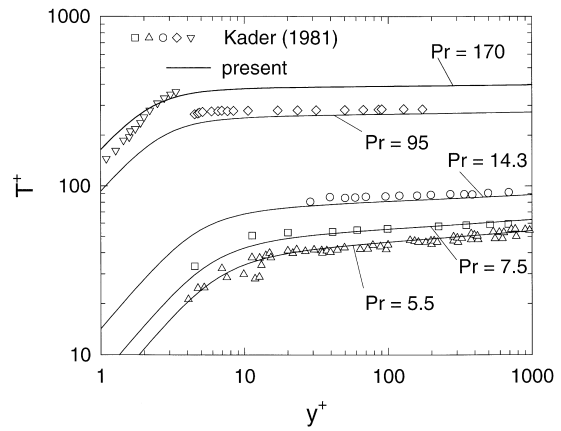


Fig. 4. Dimensionless temperature profiles for $5.5 \leq Pr \leq 170$ in pipe flow.

$Pr_t = 0.85$. The current formulation is in good agreement with the correlation of Sleicher and Rouse [19], whereas the results obtained with the van Driest proposal progressively deviate as the Pr number increases due to the $\alpha_t \propto y^{+4}$ dependency observed in Fig. 2.

Finally, the dimensionless temperature profiles reported by Kader [20] are shown in Fig. 4 together with the predictions from the present mixing length formulation using $Pr_t = 0.85$. There is reasonable agreement between the experimental and the predicted temperature profiles suggesting that the proposed change of the constant A in the original van Driest equation (3), given by Eq. (4), provides both accurate predictions for both momentum and heat transfer over a wide range of conditions.

4. Conclusions

A new mixing length equation for momentum transfer, consistent with $\nu_t \propto y^{+3}$ near the wall, has been presented. A constant $Pr_t = 0.85$ is sufficient to predict heat fluxes and temperature profiles in agreement with literature data for pipe flow and for $Pr \geq 5$.

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